Unit IV L^p-spaces, convex functions, Jensen's inequality, Holder and Minkowski inequality completeness of L^p.

The L^p spaces:

If $[X, S, \mu]$ is a measure space[1] and $p > 0$ we define $L^p(\mu)$ or $L^p(X, \mu)$ to be the class of functions $\{ f : \int |f|^p d\mu^{2} < \infty \}$.

☆ Any two functions equal almost every where specify the same element of $L^p(\mu)$.

eg On the real line if $X = (a, b)$ and μ is Lebesgue measure and S is the collection of all Lebesgue measurable sets then $[X, S, \mu]$ is a measure ~~able~~ space.

[from page 102 - 105, write all the definitions and the statement of theorems, they will be needed to understand this unit].

☆ i.e. $f \in L^p(\mu) \Rightarrow f$ is measurable and

$$\int |f|^p d\mu < \infty.$$

Definition 2 L^p - norm.

Let $f \in L^p(\mu)$ then the L^p- norm of f, denoted by $\|f\|_p$ is given by

$$\left(\int |f|^p d\mu \right)^{1/p}.$$

1 → see page numbe 102, definition 8, 9.

2) See definition 12, 13 and 14 in page 105

* (1) if f and g are measurable and $f = g$ a.e. then $\|f\|_p = \|g\|_p$.

(2) $\|f\|_p = 0$ if and only if f is the zero element of $L^p(\mu)$.

(3) $\|af\| = |a|\,\|f\|$ if a is a constant.

Theorem 1 let $f, g \in L^p(\mu)$ and let a, b be constants then $af + bg \in L^p(\mu)$.

Proof Given, $f, g \in L^p(\mu)$ and a and b are constants

Claim. $af \in L^p(\mu)$

Since $f \in L^p(\mu)$

$\Rightarrow \int |f|^p \, d\mu < \infty$.

and a is finite constant.

$\Rightarrow |a|^p \int |f|^p \, d\mu < \infty$.

$\Rightarrow \int |af|^p \, d\mu < \infty$.

$\therefore af \in L^p(\mu)$ —— (1)

Claim $f + g \in L^p(\mu)$.

given $|f+g| \le 2^p \max(|f|^p, |g|^p)$

$\le 2^p(|f|^p + |g|^p)$

PTO

since $|f|^p < \infty$ and $|g|^p < \infty$ [as $f, g \in L^p(M)$

$\Rightarrow \quad |f+g|^p < \infty$.

$\therefore \{f+g\} \in L^p(M)$ — (II)

from (I) & (II) it may be concluded that

$af + bg \in L^p(M)$.

★ The above theorem shows that $L^p(M)$ is a vector space.

<u>Definition 3</u>: If $[X, S, M]$ is a measure space we define $L^\infty(X, \mu)$ or just $L^\infty(M)$ to be the class of measurable functions $\{f : \text{ess sup} |f| < \infty\}$

and $\|f\|_\infty = \text{ess sup} |f|$

<u>Example</u> show that $L^\infty(X, M)$ is a vector space over the real numbers.

<u>Proof</u> It is enough to show that if $f, g \in L^\infty(X, \mu)$ then $af + bg \in L^\infty(X, \mu)$ $\forall a, b \in \mathbb{R}$.

~~Suy~~ Since, $f, g \in L^\infty(X, M)$

$\cancel{\|f\|_\infty <}$ ess sup $|f| < \infty$ — (I)

& ess sup $|g| < \infty$ — (II)

$Also,\quad |af + bg| \leq |a| \text{ ess sup } |f|$

$$+ |b| \text{ ess sup } |g|$$

i.e. $|af + bg| < \infty$ [from ① & ⑪]

$\Rightarrow$ ~~ess~~ $\sup [af + bg] < \infty$.

$\therefore af + bg \in L^{\infty}(X, M)$.